TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 350, Number 6, June 1998, Pages 2265–2276
S 0002-9947(98)01731-0

PERIODIC ORBITS OF THE RESTRICTED THREE-BODY PROBLEM

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Abstract. We prove, using a variational formulation, the existence of an infinity of periodic solutions of the restricted three-body problem. When the problem has some additional symmetry (in particular, in the autonomous case), we prove the existence of at least two periodic solutions of minimal period T, for every $T > 0$. We also study the bifurcation problem in a neighborhood of each closed orbit of the autonomous restricted three-body problem.

1. Introduction

We study the existence of periodic solutions of the restricted three-body problem. We are interested in the configuration studied by Sitnikov [14] and Moser [11]: we consider two mass points of equal mass $m_1 = m_2 > 0$ moving in the plane under Newton's law of attraction in the elliptic orbits such that the center of mass is at rest. We consider a third mass point moving on the line perpendicular to the plane containing the first two mass points and going through the center of mass, and we suppose that the third mass point does not influence the motion of the first two. Let z be the coordinate describing the motion of the third mass point, so that $z = 0$ corresponds to the center of mass of the first two mass points. The restricted three-body problem consists in determining z such that:

$$-\ddot{z}(t) = \frac{z(t)}{(z^2(t) + r^2(t))^{3/2}}$$

where $r(t) = r(t + 2\pi)$ is the distance from the center of mass to one of the first two mass points. For a small $\varepsilon > 0$, the function r takes the form (see Moser [11]):

$$r(t) = \tfrac{1}{2}(1 - \varepsilon\cos(t)) + O(\varepsilon^2).$$

In this paper, we suppose that the function $r\colon R \to R$ is continuous and T-periodic, $T > 0$. We prove, using a variational formulation, the existence of an infinity of periodic solutions of the restricted three-body problem. When the T-periodic function r satisfies the following hypothesis

$$\forall t \in [0, T/2], \quad r^2(t) = r^2(T - t)$$

(in particular, in the autonomous case) we prove that for all $k \in N$ the restricted three-body problem has at least two periodic solutions of minimal period kT. Finally, for $|\varepsilon| \neq 0$ small enough, we can see the restricted three-body problem as

Received by the editors July 20, 1995.
1991 *Mathematics Subject Classification.* Primary 34A34; Secondary 34A47.

a perturbed system of the autonomous system ($\varepsilon = 0$). We prove that in the neighborhood of each orbit:

$$Z_k = \{\theta * x_k = x_k(.+\theta) \colon \theta \in [0, kT]\}$$

where x_k is a periodic solution of minimal period (kT) of the autonomous restricted three-body problem, there are at least two (kT)-periodic solutions of the perturbed restricted three-body problem for a $|\varepsilon| \neq 0$ small enough.

The existence of an infinity of periodic solutions of the restricted three-body problem was proved for the first time by Moser [11]. The method used by Moser is different from ours. He uses the Bernouilli shift and the symbolic dynamics.

2. Variational formulation

We consider the restricted three-body problem:

$$\begin{cases} -\ddot{z}(t) = \dfrac{z(t)}{(z^2(t) + r^2(t))^{3/2}}, \\ z(0) - z(T) = \dot{z}(0) - \dot{z}(T) = 0, \end{cases} \tag{1}$$

where $T > 0$ and $r \colon R \to R$ is a continuous T-periodic function:

$$r(t+T) = r(t), \quad \forall t \in R.$$

We denote by H_T^1 the Hilbert space

$$H_T^1 = \{z \in H^1([0,T], R) | z(0) = z(T)\}$$

equipped with its usual norm

$$\|z\|^2 = \int_0^T (|\dot{z}(t)|^2 + |z(t)|^2)\, dt$$

and we define the functional f by

$$\begin{aligned} &H_T^1 \to R, \\ f \colon &z \mapsto \int_0^T \left(\frac{1}{2}|\dot{z}(t)|^2 + \frac{1}{\sqrt{z^2(t) + r^2(t)}} \right) dt. \end{aligned}$$

It is not difficult to prove that f is of class C^2 and that the solutions of the system (1) are the critical points of the functional f.

We denote by $\mathcal{Z}(f)$ the set of the critical points of f:

$$\mathcal{Z}(f) = \{z \in H_T^1 | f'(z) = 0\}.$$

The Morse index of $z \in \mathcal{Z}(f)$ is defined to be the supremum of the dimensions of the subspaces of H_T^1 on which $f''(z)$ is negative definite. The nullity of $z \in \mathcal{Z}(f)$ is defined as the dimension of $\ker f''(z)$. We denote by $i(z)$ and $\nu(z)$ respectively, the Morse index and the nullity of the critical point z.

Let

$$m_0 = \left(\max_{t \in [0,T]} |r(t)| \right)^{3/2}$$

and

$$k_1 = \max \left\{ k \in N; k < \frac{T}{2\pi m_0} \right\}.$$

Theorem 1. *If $k_1 \geq 1$, then*

(a) *The functional f has at least $4k_1$ of nontrivial critical points, and if x_k is a critical point then $-x_k$ is also a critical point.*

(b) *For all k, $1 \leq k \leq 2k_1$, there exist two critical points x_k, y_k of f such that*

$$i(x_k) \leq k$$

and

$$i(y_k) + \nu(y_k) \geq k.$$

Corollary 1. *There exist $p_0 \in N$ and an increasing sequence of integers $(k_p)_{p=1}^{\infty}$ such that*

$$\lim_{p\to\infty} k_p = +\infty$$

and, for all $p \geq p_0$, the restricted three-body problem has at least $4k_p$ nontrivial (pT)-periodic solutions.

Proof of Corollary 1. The function r is T-periodic. Then, for all $p \in N$, $p \geq 1$, it is also a (pT)-periodic function. Let

$$k_p = \max\left\{k \in N; k < \frac{pT}{2\pi m_0}\right\}.$$

It is clear that $(k_p)_{p=1,\infty}$ is an increasing sequence, that there exists $p_0 \in N$ such that $k_p \geq 1$ for $p \geq p_0$ and $\lim_{p\to\infty} k_p = +\infty$. Now, apply Theorem 1 with the period (pT). □

Corollary 2. *The restricted three-body problem has an infinity of distinct closed orbits.*

Proof of Theorem 1. We say that f satisfies the Palais-Smale condition, shortly (P.S.) condition, at the level $c \in R$, if, for every sequence $(z_k) \subset H_T^1$ such that

$$f(z_k) \to c \text{ and } f'(z_k) \to 0, \quad \text{as } k \to \infty$$

there is a convergent subsequence. □

Lemma 1. (a) *The functional f is even and satisfies the $(P.S.)$ condition at every level $c > 0$.*

(b) *$Z(f)$ is a compact set.*

Proof. (a) It is clear that f is even. Let $(z_k) \subset H_T^1$ be a sequence such that

$$f'(z_k)h \to 0, \quad \forall h \in H_T^1, \text{ as } k \to \infty \tag{2.1}$$

and

$$f(z_k) \to c > 0, \quad \text{as } k \to \infty. \tag{2.2}$$

We denote

$$E_0 = \left\{x \in H_T^1 | \int_0^T x(s)\,ds = 0\right\}.$$

It is not difficult to see that $H_T^1 = E_0 \oplus R$. Let $(x_k) \subset E_0$ and $(\xi_k) \subset R$ be such that

$$z_k = x_k + \xi_k, \quad \forall k \in N.$$

Let us prove that (z_k) is bounded in H_T^1. Using (2.2), we can see that $(\dot{z}_k) = (\dot{x}_k)$ is bounded in $L^2([0,T])$. Then the sequence (x_k) is bounded in H_T^1 and $L^\infty([0,T])$. Hence, (z_k) is bounded in H_T^1 if and only if (ξ_k) is bounded in R. Suppose that (ξ_k) is not bounded in R. Then, up to a subsequence, we can suppose that:

$$\xi_k \to +\infty, \quad \text{as } k \to \infty.$$

The sequence (x_k) is bounded in H_T^1; then

$$f'(z_k)x_k = \int_0^T |\dot{z}_k|^2 - \frac{(x_k+\xi_k)x_k}{((x_k+\xi_k)^2+r^2)^{3/2}} \to 0 \quad \text{as } k \to \infty.$$

On the other hand,

$$\int_0^T \frac{(x_k+\xi_k)x_k}{((x_k+\xi_k)^2+r^2)^{3/2}} \to 0 \quad \text{as } k \to \infty$$

because (x_k) is bounded in L^∞ and $\xi_k \to +\infty$. Then

$$\int_0^T |\dot{z}_k|^2 \to 0, \quad \text{as } k \to \infty.$$

For the same reason,

$$\int_0^T \frac{1}{\sqrt{(x_k+\xi_k)^2+r^2}} \to 0, \quad \text{as } k \to \infty.$$

This implies that

$$f(z_k) \to 0, \quad \text{as } k \to \infty$$

which is a contradiction with $c > 0$.

Thus, (z_k) is bounded in H_T^1. Then, up to subsequence, (z_k) converges weakly in H_T^1 and strongly in $L^\infty([0,T])$ to $z \in H_T^1$. We have

$$\lim_{k\to\infty} f'(z_k)z = 0 = f'(z)z,$$

and

$$\lim_{k\to\infty} f'(z_k)z_k = \lim_{k\to\infty} \int_0^T |\dot{z}_k|^2 - \frac{z_k^2}{(z_k^2+r^2)^{3/2}} = 0.$$

Then

$$\lim_{k\to\infty} \int_0^T |\dot{z}_k|^2 = \int_0^T |\dot{z}|^2$$

and

$$\lim_{k\to\infty} \int_0^T \dot{z}_k\dot{z} = \int_0^T |\dot{z}|^2.$$

Hence, $(\dot{z}_k)$ converges to $\dot{z}$ in $L^2([0,T])$. Then (z_k) converge to z in H_T^1.

(b) Let (z_k) be a sequence of $\mathcal{Z}(f)$. Let $(\xi_k) \in R$, $(x_k) \in E_0$ be such that

$$\forall k \in N, \quad z_k = x_k + \xi_k.$$

It is not difficult to prove that $(\dot{z}_k) = (\dot{x}_k)$ is bounded in $L^2([0,T],R)$ and $(f(z_k))$ is bounded in R. Then, up to a subsequence, we can suppose that $(f(z_k))$ converges to c in R. If $c > 0$, by the (P.S.) condition at the level c, (z_k) has a convergent subsequence. Let us prove that $c > 0$. Arguing by contradiction, we suppose that $c = 0$. Then, (x_k) converges strongly in H_T^1 and in L^∞ to 0 and $(|\xi_k|)$ goes to $+\infty$

as $k \to \infty$. On the other hand, for all $k \in N$, z_k is a T-periodic solution of the restricted three-body problem; hence

$$0 = \int_0^T \frac{(x_k(t) + \xi_k)}{((x_k + \xi_k)^2 + r^2(t))^{3/2}} dt, \quad \forall k \in N.$$

This implies that

$$0 = |\xi_k|\xi_k \int_0^T \frac{(x_k(t) + \xi_k)}{((x_k + \xi_k)^2 + r^2(t))^{3/2}} dt, \quad \forall k \in N$$

which is a contradiction with $T > 0$, because, when k goes to ∞ the term on the right-hand side goes to T.

Let Σ denote the family of sets $A \subset H_T^1 - \{0\}$ such that A is closed in H_T^1 and symmetric with respect to 0. For $A \in \Sigma$, we define the cogenus of A to be

$$\gamma^-(A) = \inf\{n \in N | \exists \phi \colon A \to S^{n-1} \text{ odd and continuous}\}$$

and the genus to be

$$\gamma^+(A) = \sup\{n \in N | \exists \phi \colon S^{n-1} \to A \text{ odd and continuous}\}.$$

When there does not exist such a ϕ, we set $\gamma^\pm(A) = \infty$.

The proof of the following properties can be found in [12], [5], [6]: $\square$

Proposition 1. 1°. *If $x \neq 0$, $\gamma^\pm(\{x\} \cup \{-x\}) = 1$.*

2°. *If there is an odd map $f \in C(A, B)$, then $\gamma^\pm(A) \leq \gamma^\pm(B)$.*

3°. *$\forall A \in \Sigma$, $\gamma^+(A) \leq \gamma^-(A)$.*

4°. *If X is a subspace of H_T^1 of dimension k and $S = \{x \in X | \, \|x\| = r\}$, $r > 0$, then $\gamma^\pm(S) = k$.*

5°. *If X is a subspace of H_T^1 of codimension k and $A \in \Sigma$ with $\gamma^-(A) > k$, then $A \cap X \neq \varnothing$.*

For all $k \in N$, we let

$$\gamma_k^\pm = \{A \in \Sigma;\ A \text{ is compact},\ \gamma^\pm(A) \geq k\}$$

and

$$c_k^\pm = \inf_{A \in \gamma_{k+1}^\pm} \max_{x \in A} f(x).$$

By property 3 of Proposition 1, for all $k \in N$, we have

$$c_k^- \leq c_k^+$$

and it is clear that

$$\gamma_1^\pm \supset \gamma_2^\pm \supset \cdots \supset \gamma_k^\pm \supset \cdots$$

so we also have

$$c_0^\pm \leq c_1^\pm \leq c_2^\pm \leq \cdots \leq c_k^\pm \leq \cdots .$$

On the other hand, if f satisfies the Palais-Smale condition at the level $c = c_k^\pm$, it is known that $c = c_k^\pm$ is a critical value of f; if $c = c_k^- = c_{k+1}^- < f(0)$ and f satisfies the Palais-Smale condition at the level c, then $f^{-1}(\{c\})$ contains infinitely many distinct critical points (see [2], [7] and [13]). Hence, to prove Theorem 1, we must prove the existence of $c_k^\pm$ such that $0 < c_k^\pm < f(0)$, since f satisfies the Palais-Smale condition at every level $c > 0$.

Proposition 2. (1) $0 = c_0^{\pm} < c_1^{\pm}$.

(2) $c_{2k_1}^{\pm} < f(0)$.

Proof. (1) Let us prove that $c_0^{\pm} = 0$. For all $x \in H_T^1$, $x \neq 0$, we have $\gamma^{\pm}(\{x, -x\}) = 1$. Then $c_0^{\pm} = \inf\{f(x); x \in H_T^1\}$ because

$$f(x) = \max_{t \in \{x,-x\}} f(t), \quad \forall x \in H_T^1.$$

On the other hand, we remark that $\inf\{f(x); x \in H_T^1\} = 0$ because f is a positive functional and $f(\xi) \to 0$ as $|\xi| \to +\infty$, if $\xi \in R$.

Let us prove that $c_1^- > 0$. We denote

$$\delta = \inf_{x \in E_0} f(x).$$

Using property 5 of Proposition 1, for all $A \in \gamma_2^-$, we have $A \cap E_0 \neq \varnothing$, so that

$$c_1^- \geq \delta.$$

Let (x_k) be a minimizing sequence of f over E_0: $f(x_k) \to \delta$ when $k \to \infty$. If $\delta = 0$, then (x_k) converges to zero in H_T^1 and in $L^\infty([0,T])$ so that

$$\lim_{k\to\infty} f(x_k) = \lim_{k\to\infty} \int_0^T \left(\frac{1}{2}|\dot{x}_k(t)|^2 + \frac{1}{\sqrt{x_k(t)^2 + r(t)^2}} \right) dt = \int_0^T \frac{1}{|r(t)|} dt > 0$$

which is a contradiction with $\delta = 0$. Thus, $\delta > 0$ and $c_1^+ \geq c_1^- > 0$.

(2) It is clear that 0 is a critical point of f. Let X be the subspace of H_T^1 generated by the following functions:

$$1, \sin\left(\frac{2\pi}{T}t\right), \cos\left(\frac{2\pi}{T}t\right), \sin\left(\frac{4\pi}{T}t\right), \cos\left(\frac{4\pi}{T}t\right), \ldots, \sin\left(\frac{2k_1\pi}{T}t\right), \cos\left(\frac{2k_1\pi}{T}t\right).$$

It is clear that the dimension of X is equal to $2k_1 + 1$ and that there exists $\alpha > 0$ such that

$$\max_{x \in S} f(x) < f(0)$$

where $S = \{x \in X \mid \|x\| = \alpha\}$ because $f''(0)v.v < 0$ for all $v \in X$. Using Proposition 1, we have $\gamma^{\pm}(S) = 2k_1 + 1$ and by definition of the $c_k^{\pm}$'s:

$$c_{2k_1}^{\pm} \leq \max_{x \in S} f(x) < f(0).$$

The proof of (a) of Theorem 1 follows from Proposition 2. To prove (b) of Theorem 1, we need the following proposition (see [3], [4], Proposition 6.8, page 170, [9]): $\square$

Proposition 3. *Let Ω be a C^2 open subset of a Hilbert space H and let $f \in C^2(\Omega, R)$. Assume f' is a Fredholm operator (of null index) on the critical set $\mathcal{Z}(f) = \{u \in \Omega, f'(u) = 0\}$. Suppose furthermore that $\mathcal{Z}(f)$ is compact. Then, for any $\varepsilon > 0$ and $\eta > 0$, there exists $g_\varepsilon \in C^2(\Omega, R)$ satisfying the following properties:*

(g.1) *$g_\varepsilon(u) = f(u)$, if $\|u - \mathcal{Z}(f)\| \geq \eta$.*

(g.2) *$|g_\varepsilon(u) - f(u)| \leq \varepsilon$, $\|g'_\varepsilon(u) - f'(u)\| \leq \varepsilon$, $\forall u \in \Omega$.*

(g.3) *$\|g''_\varepsilon(u) - f''(u)\|_{\mathcal{L}(H,H')} \leq \varepsilon$, $\forall u \in \Omega$.*

(g.4) *The critical points of g_ε (if any) are in finite number and are nondegenerate.*

(g.5) *g'_ε is a Fredholm operator (of null index) and if f satisfies the (P.S.) condition, then g_ε can also be chosen so that it satisfies the (P.S.) condition.*

It is not difficult to prove that the functional f corresponding to the restricted three-body problem satisfies the assumptions of Proposition 3. Hence, we construct a sequence of functionals (g_n) such that g_n satisfies properties (g.1)–(g.5) of Proposition 3 with $\varepsilon = \eta = \frac{1}{n}$ and we can suppose that g_n is even for all n (if not, we replace $g_n(.)$ by $(g_n(.) + g_n(-.))/2$).

For all $n \in N$, we set

$$(c_n)_k^{\pm} = \inf_{A \in \gamma_{k+1}^{\pm}} \max_{x \in A} g_n(x)$$

by (g.2) of Proposition 3; it is clear that

$$c_k^{\pm} - 1/n \le (c_n)_k^{\pm} \le c_k^{\pm} + 1/n, \quad \forall k \in N.$$

Then, for n large enough, $(c_n)_k^{\pm}$ $(k = 1, 2, \dots, 2k_1)$ are critical values of g_n. In [5] (see also [6]) Bahri and Lions proved the existence of at least two critical points of g_n, x_k^n and y_k^n, for $k = 1, 2, \dots, 2k_1$, critical points of g_n such that

$$(c_n)_k^{+} = g_n(y_k^n), \quad i_n(y_k^n) + \nu_n(y_k^n) \ge k,$$

and

$$(c_n)_k^{-} = g_n(x_k^n), \quad i_n(x_k^n) \le k,$$

where i_n and ν_n are, respectively, the Morse index and the nullity of x_k^n and y_k^n, as critical points of the functional g_n. Now, using Proposition 3 and the Palais-Smale condition, we can prove (b) of Theorem 1.

Remark 1. (a) It is not difficult to see that, for any T-periodic solution x of the restricted three-body problem, the dimension of the space of solutions of the linearized system

$$\text{(2)} \qquad \begin{cases} \ddot{y}(t) = \dfrac{2x^2(t) - r^2(t)}{(x^2(t) + r^2(t))^{5/2}} y(t), \\ y(0) - y(T) = \dot{y}(0) - \dot{y}(T) = 0 \end{cases}$$

is equal to the nullity of the critical point x of f. Then the nullity of any critical point x of f satisfies $0 \le \nu(x) \le 2$.

(b) In the autonomous case, $(r(.) = r_0)$, if x is a T-periodic solution of the restricted three-body problem, then $y = \dot{x}$ is a solution of the linearized problem (2). Thus, $\nu(x) \ge 1$. Using a theorem due to Willem [15] (see also [10], page 228), we can prove that the nullity of any critical point x of f is equal to one.

3. Variational formulation with a symmetry condition on r

In this section, we assume that the T-periodic function r satisfies the following symmetry condition (see [8] for another application):

$$\forall t \in [0, T/2], \quad r^2(t) = r^2(T - t).$$

We consider the system

$$(3)_p \qquad \begin{cases} -\ddot{u}(t) = \dfrac{u(t)}{(u^2(t) + r^2(t))^{3/2}}, & t \in [0, pT/2], \\ u(0) = u(pT/2) = 0. \end{cases}$$

We denote by E_p, $p \in N^*$, the Hilbert space

$$E_p = \{u \in H^1([0, pT/2], R) | u(0) = u(pT/2) = 0\}$$

equipped with its usual norm

$$\|u\|^2 = \int_0^{pT/2} |\dot{u}(t)|^2 dt$$

and we define the functional f_p by

$$\begin{aligned} &E_p \to R, \\ &f_p \colon u \mapsto \int_0^{pT/2} \left(\frac{1}{2}|\dot{u}(t)|^2 + \frac{1}{\sqrt{u^2(t) + r^2(t)}} \right) dt. \end{aligned}$$

It is not difficult to prove that f_p is of class C^1 and that the solutions of system $(3)_p$ are the critical points of the functional f_p.

Theorem 2. *For all $p \in N$, let m_0 and k_p be such that*

$$m_0 = \left(\max_{t \in [0, T/2]} |r(t)| \right)^{3/2}, \quad k_p = \max \left\{ k \in N; k < \frac{pT}{2\pi m_0} \right\}.$$

If $k_p \geq 1$, the functional f_p has at least two critical points $u_p, -u_p$ such that

$$u_p(t) \neq 0, \quad \forall t \in]0, pT/2[. \tag{2.3}$$

Corollary 3. *There exists $p_0 \in N$ such that, for all $p \in N$, $p \geq p_0$, the restricted three-body problem has at least two nontrivial periodic solutions, $x_p, -x_p$, of minimal period (pT) such that:*

$$x_p(t + pT/2) = -x_p(pT/2 - t), \quad \forall t \in [0, pT/2].$$

Proof of Corollary 3. The sequence (k_p) of Theorem 2 is an increasing sequence such that

$$\lim_{p \to \infty} k_p = +\infty.$$

Then, there exists p_0 such that, for all $p \in N$, $p \geq p_0$, $k_p \geq 1$. Starting from the critical point u_p of f_p, we construct a (pT)-periodic solution of the restricted three-body problem by putting

$$x_p(t) = \begin{cases} u_p(t), & t \in [0, pT/2], \\ -u_p(pT - t), & t \in [pT/2, pT], \end{cases}$$

and then extending x_p by (pT)-periodicity, to the whole real line. It is not difficult to see that x_p is a (pT)-periodic solution of the restricted three-body problem and it is clear that the minimal period of x_p is (pT) because u_p satisfies (2.3). $\square$

Proof of Theorem 2.

Lemma 2. *The functional f_p is even, bounded from below in E_p and satisfies the $(P.S.)$ condition at every level $c \in R$.*

Proof of Lemma 2. It is clear that f_p is even and bounded from below. Let $(z_k) \subset E_p$ be a sequence such that:

$$f_p'(z_k)h \to 0, \quad \forall h \in E_p, \quad \text{as } k \to \infty \tag{2.4}$$

and

$$f_p(z_k) \to c \in R, \quad \text{as } k \to \infty. \tag{2.5}$$

Then (z_k) is bounded in E_p. Up to a subsequence, (z_k) converges weakly in E_p and strongly in $L^\infty([0,pT/2])$ to $z \in E_p$. We have

$$\lim_{k\to\infty} f_p'(z_k)z = 0 = f_p'(z)z,$$

and

$$\lim_{k\to\infty} f_p'(z_k)z_k = \lim_{k\to\infty} \int_0^{pT/2} |\dot z_k|^2 - \frac{z_k^2}{(z_k^2+r^2)^{3/2}} = 0.$$

Then

$$\lim_{k\to\infty} \int_0^{pT/2} |\dot z_k|^2 = \int_0^{pT/2} |\dot z|^2$$

and

$$\lim_{k\to\infty} \int_0^{pT/2} \dot z_k \dot z = \int_0^{pT/2} |\dot z|^2.$$

Hence, $(\dot z_k)$ converges to $\dot z$ in $L^2([0,pT/2])$. Then (z_k) converges to z in E_p. □

Lemma 3. *There exists $p_0 \in N$ such that, for all $p \geq p_0$, the minimum of f_p is achieved in $u_p \in E_p$ such that*

$$0 < f_p(u_p) = \min\{f_p(u); u \in E_p\} < f_p(0), \quad \forall p \geq p_0.$$

Proof of Lemma 3. With the (P.S.) condition, it is easy to see that the minimum of the functional f_p is achieved. For all $p \in N$, let u_p be such that

$$f_p(u_p) = \min\{f_p(u); u \in E_p\}. \tag{2.6}$$

If $k_p \geq 1$, we define $w_p \in E_p$ as follows:

$$w_p(t) = \sin\left(\frac{2\pi}{pT}k_p t\right), \quad \forall t \in [0, pT/2].$$

We have

$$\begin{aligned} f_p''(0)w_p w_p &= \int_0^{pT/2} (\dot w_p(t))^2 dt - \int_0^{pT/2} \frac{(w_p(t))^2}{r^3(t)} dt \\ &\leq \left(\left(\frac{2\pi}{pT}k_p\right)^2 - \frac{1}{m_0^2}\right) pT/4 \\ &< 0. \end{aligned}$$

Then

$$f_p(u_p) < f_p(0)$$

and it is not difficult to see that

$$f_p(u) > 0, \quad \forall p, \forall u.$$

By Lemma 2 and Lemma 3, we have the existence of a critical point u_p of the functional f_p. To prove (2.4), we can suppose that

$$u_p(t) \geq 0, \quad t \in \left[0, p\tfrac{T}{2}\right] \tag{2.7}$$

because $|u_p|$ is also in E_p and $f_p(|u_p|) = f_p(u_p)$. On the other hand, u_p is a critical point of f_0; thus u_p satisfies the system $(3)_p$. Now, using $(3)_p$ and (2.7), we deduce (2.4). □

4. Bifurcation in the restricted three-body problem

In this section, we are interested in the case $T = 2\pi$ and $r = r_\varepsilon$, $\varepsilon \in R$, is given by Moser [11]:

$$r_\varepsilon(t) = \tfrac{1}{2}(1 - \varepsilon\cos(t)) + O(\varepsilon^2)$$

and we suppose that $r = r_\varepsilon$ is C^∞ in ε and t.

For all $p \geq 1$ and for $|\varepsilon| \neq 0$ small enough, we can see the restricted three-body problem

$$(1)_\varepsilon^p \qquad \begin{cases} -\ddot{z}(t) = \dfrac{z(t)}{(z^2(t) + r_\varepsilon^2(t))^{3/2}}, \\ z(0) - z(2p\pi) = \dot{z}(0) - \dot{z}(2p\pi) = 0 \end{cases}$$

as a perturbed system of the autonomous restricted three-body problem for $\varepsilon = 0$:

$$(1)_0^p \qquad \begin{cases} -\ddot{z}(t) = \dfrac{z(t)}{(z^2(t) + \frac{1}{4})^{3/2}}, \\ z(0) - z(2p\pi) = \dot{z}(0) - \dot{z}(2p\pi) = 0. \end{cases}$$

Proposition 4. *For all $p \geq 1$, the autonomous restricted three-body problem $(1)_0^p$ has at least two nontrivial periodic solutions $x_p, -x_p$ of minimal period $2p\pi$.*

Proof. For all $p \in N$, $p \geq 1$, we have

$$m_0 = \left(\max_{t\in[0,\pi]} |r_0(t)|\right)^{3/2} = \frac{1}{\sqrt{8}},$$
$$k_p = \max\{k \in N; k < p\sqrt{8}\}.$$

Then $k_p \geq 1$ for all $p \geq 1$. By Theorem 2 and Corollary 3, we have Proposition 4. □

Remark 2. The system $(1)_0^p$ is autonomous. If x is a solution of $(1)_0^p$, then, for all $\tau \in [0, 2p\pi[$, $\tau * x = x(.+\tau)$ is also a solution. Hence, we have an orbit of solutions:

$$Z(x) = \{\tau * x = x(.+\tau) \colon \tau \in [0, 2p\pi]\}.$$

Theorem 3. *There exists $\overline{\varepsilon} > 0$ such that: for all ε, $0 < |\varepsilon| < \overline{\varepsilon}$, the perturbed restricted three-body problem $(1)_\varepsilon^p$ has at least two $(2p\pi)$-periodic solutions near the orbit $Z(x_p)$, where x_p is the $(2p\pi)$-periodic solution given by Proposition 4.*

Theorem 3 is a particular case of a general theorem of [1]. Here, we use the implicit function theorem and the following lemmas:

Lemma 4. *Let $a\colon R \to R$, $f\colon R \to R$ be two T-periodic functions. Let E_0 be the space of solutions of the linear system:*

$$\ddot{y}(t) = a(t)y(t), \quad y(0) - y(T) = \dot{y}(0) - \dot{y}(T) = 0.$$

Then the linear system

$$\ddot{y}(t) = a(t)y(t) + f(t), \quad y(0) - y(T) = \dot{y}(0) - \dot{y}(T) = 0$$

has a solution if and only if $\int_0^T f(t)v(t)dt = 0$ for all $v \in E_0$.

Lemma 4 is a classical result for the problem with periodic boundary conditions. We define the function $F\colon R \times H^1_{2p\pi} \to H^1_{2p\pi}$ by

$$\langle F(\varepsilon, z), h\rangle = \int_0^{2p\pi} \left(\dot{z}(t)\dot{h}(t) - \frac{z(t)h(t)}{(z^2(t) + r_\varepsilon^2(t))^{3/2}} \right) dt, \quad \forall h \in H^1_{2p\pi}$$

where $\langle ., .\rangle$ is the scalar product in $H^1_{2p\pi}$. It is not difficult to prove that F is C^∞ and

$$F(\varepsilon, z) = 0 \Leftrightarrow z \text{ is a solution of } (1)^p_\varepsilon.$$

We denote by $\mathcal{L}_\tau = F'_z(0, \tau * x_p)$ and E_τ the subspace of $H^1_{2p\pi}$ generated by $y = \tau * \dot{x}_p$.

Remark 3. By Remark 2 and Lemma 4, we can deduce that

$$\operatorname{Ker}(\mathcal{L}_\tau) = E_\tau, \text{ and } \operatorname{Range}(\mathcal{L}_\tau) = E_\tau^\perp.$$

Lemma 5. *There exist* $\varepsilon_0 > 0$ *and two* C^∞ *functions:*

$$\lambda :]-\varepsilon_0, \varepsilon_0[\times R \to R, \quad z :]-\varepsilon_0, \varepsilon_0[\times R \to H^1_{2p\pi}$$

such that

(a) $z(0, \tau) = \tau * x_p$, $z(\varepsilon, \tau + 2p\pi) = z(\varepsilon, \tau)$, $\forall \tau \in R, \forall \varepsilon \in]-\varepsilon_0, \varepsilon_0[$,
(b) $F(\varepsilon, z(\varepsilon, \tau)) + \lambda(\varepsilon, \tau)\tau * \dot{x}_p = 0$, $\forall \tau \in R$, $\forall \varepsilon \in]-\varepsilon_0, \varepsilon_0[$,
(c) $z(\varepsilon, \tau_1)(.) \neq z(\varepsilon, \tau_2)(.)$, $\forall \tau_1, \tau_2 \in [0, 2p\pi[, \tau_1 \neq \tau_2, \forall \varepsilon \in]-\varepsilon_0, \varepsilon_0[$.

Proof of Lemma 5. Let $G\colon R \times R \times R \times H^1_{2p\pi} \to R \times H^1_{2p\pi}$ be defined by

$$G(\varepsilon, \tau, \lambda, z) = (\langle z, \tau * \dot{x}_p\rangle, F(\varepsilon, z) + \lambda\tau * \dot{x}_p).$$

Using Remark 3, it is not difficult to see that

$$G(0, \tau, 0, \tau * x_p) = 0 \text{ and } G'_{(\lambda,z)}(0, \tau, 0, \tau * x_p) \text{ is an isomorphism}$$

for all $\tau \in [0, pT]$. Then we can apply the implicit function theorem for the function G in the point $(0, \mu, 0, \mu * x_p)$ for all $\mu \in [0, 2p\pi]$: there exist $\varepsilon_\mu > 0$, an open neighborhood $V(\mu)$ of μ and a unique pair of C^∞ functions

$$\lambda_\mu :]-\varepsilon_\mu, \varepsilon_\mu[\times V(\mu) \to R, \quad z_\mu :]-\varepsilon_\mu, \varepsilon_\mu[\times V(\mu) \to H^1_{2p\pi}$$

such that

$$0 = G(\varepsilon, \tau, \lambda_\mu(\varepsilon, \tau), z_\mu(\varepsilon, \tau)), \quad \forall (\varepsilon, \tau) \in]-\varepsilon_\mu, \varepsilon_\mu[\times V(\mu).$$

Now, using the $2p\pi$ periodicity of G in τ, the unicity of the functions λ_μ, z_μ and the compactness of $[0, 2p\pi]$, we can find $\varepsilon_0 > 0$ and construct two functions λ, z that satisfy (a) and (b) of Lemma 5. Let $\tau_1 \neq \tau_2 \in R$ be such that $0 \leq \tau_1 < \tau_2 < 2p\pi$. It is clear that $\tau_1 * x_p \neq \tau_2 * x_p$, because $0 < \tau_2 - \tau_1 < 2p\pi$ and $2p\pi$ is the minimal period of x_p. On the other hand, z is a continuous function of ε and τ; then we can choose $\varepsilon_0 > 0$ such that (c) is true. □

Proof of Theorem 3. Let ε be such that $0 < |\varepsilon| < \varepsilon_0$. We define the function $h_\varepsilon\colon R \to R$ by

$$h_\varepsilon(\tau) = \int_0^{2p\pi} \left(\frac{1}{2}|\dot{z}(t)|^2 + \frac{1}{\sqrt{z^2(t) + r_\varepsilon^2(t)}} \right) dt$$

where $z = z(\varepsilon, \tau)$ is the function given by Lemma 5. Using Lemma 5, it is not difficult to see that h_ε is a C^∞ $2p\pi$-periodic function and

$$h'_\varepsilon(\tau) = \langle F(\varepsilon, z(\varepsilon,\tau)), \frac{\partial z(\varepsilon,\tau)}{\partial \tau}\rangle = -\lambda(\varepsilon,\tau)\langle \tau * \dot{x}_p, \frac{\partial z(\varepsilon,\tau)}{\partial \tau}\rangle$$

and

$$\langle \tau * \dot{x}_p, \frac{\partial z(0,\tau)}{\partial \tau}\rangle = \langle \tau * \dot{x}_p, \tau * \dot{x}_p\rangle \neq 0;$$

then the continuity of $\frac{\partial z}{\partial r}$ in ε implies that there exists $0 < \overline{\varepsilon} \leq \varepsilon_0$ such that

$$\langle \tau * \dot{x}_p, \frac{\partial z(\varepsilon,\tau)}{\partial \tau}\rangle \neq 0, \quad \forall \varepsilon, |\varepsilon| < \overline{\varepsilon}.$$

Hence, for all $\varepsilon \in]-\overline{\varepsilon}, \overline{\varepsilon}[$, if $\overline{\tau}$ is a critical point of h_ε, then $F(\varepsilon, z(\varepsilon, \overline{\tau})) = 0$.

On the other hand, h_ε achieves its maximum and minimum in two points $\tau_1 \neq \tau_2 \in [0, 2p\pi[$ because h_ε is continuous and $2p\pi$-periodic. Then $z(\varepsilon, \tau_i)$, $i = 1, 2$, are two $2p\pi$-periodic solutions of $(1)^p_\varepsilon$ and by the continuity of z in ε, we can see that the two solutions are in a neighborhood of the orbit $Z(x_p)$. □

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